MATHEMATICS

B.Sc (Part-I), Paper-II

Differential Calculus

by

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Topic - Indeterminate Forms

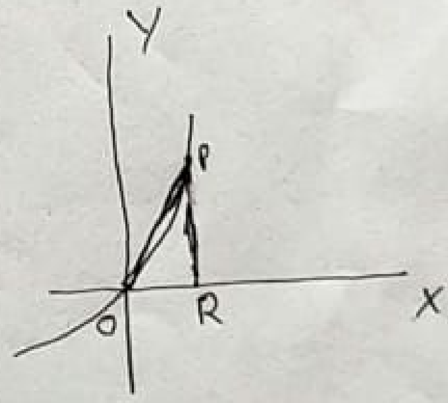
Definition :- Let $f(x)$ and $\phi(x)$ be two functions of a single variable x and $f(x)$ and $\phi(x)$ both tend to zero as $x \to 0$, then $\underset{x\to 0}{Lt} \frac{f(x)}{\phi(x)}$ will be of the form $\frac{0}{0}$ which is meaningless. The form $\frac{0}{0}$ is called the indeterminate form. There are other types of indeterminate forms as $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞ etc.

(I) The form $\frac{0}{0}$:—

Let us consider a curve passing through the origin and defined by the equations

$$\left.\begin{array}{l} x = \psi(t) \\ y = \phi(t) \end{array}\right\}$$

as shown in the figure



Let P be a point on the curve whose coordinates w.r.t x-axis and y-axis be (x, y), near the origin and let a be the value of t corresponding to the origin so that

$$\phi(a) = 0 \text{ and } \psi(a) = 0$$

Since P is ~~the~~ very near to the origin on the curve, then we have ultimately

$$Lt \frac{y}{x} = Lt \tan POR = \underset{x\to 0}{Lt} \frac{dy}{dx} = \underset{t\to a}{Lt} \frac{\phi'(t)}{\psi'(t)}$$

where dash denotes the differentiation.

Thus $\displaystyle \underset{t\to a}{Lt}\frac{\phi(t)}{\psi(t)} = \underset{t\to a}{Lt}\frac{\phi'(t)}{\psi'(t)}$

Here if $\dfrac{\phi'(t)}{\psi(t)}$ is not of the form $\frac{0}{0}$ when t takes a value a, we then obtain

$$\underset{t\to a}{Lt}\frac{\phi(t)}{\psi(t)} = \frac{\phi'(a)}{\psi'(a)}$$

Again if $\dfrac{\phi'(t)}{\psi'(t)}$ is also of the form $\frac{0}{0}$, we may repeat the process and hence

$$\underset{t\to a}{Lt}\frac{\phi'(t)}{\psi'(t)} = \underset{t\to a}{Lt}\frac{\phi''(t)}{\psi''(t)} = \text{etc.}$$

proceeding in this manner until we reach at a definite value when a is substituted for t.

i.e. we shall differentiate numerator and denominator in the same way separately until we get a definite value.

(II) The form $\frac{\infty}{\infty}$:-

If $\underset{x\to a}{Lt} f(x)$ and $\underset{x\to a}{Lt}\phi(x)$ be both infinite, then

$$\underset{x\to a}{Lt}\frac{f(x)}{\phi(x)} = \underset{x\to a}{Lt}\frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}} \text{ which is of the form } \frac{0}{0}$$

Since $\phi(x)$ & $f(x)$ are infinite so $\left[\frac{1}{\phi(x)} \text{ and } \frac{1}{f(x)} \text{ are both zero}\right]$

$$\underset{x\to a}{Lt}\frac{f(x)}{\phi(x)} = \underset{x\to a}{Lt}\frac{\frac{\phi'(x)}{[\phi(x)]^2}}{\frac{f'(x)}{[f(x)]^2}} = \underset{x\to a}{Lt}\left[\frac{\phi(x)}{f'(x)}\right]^{-2}\frac{\phi'(x)}{f'(x)} \quad \text{....(A)}$$

Now there arises three cases as below.

Case I. If $\underset{x\to a}{Lt}\dfrac{f(x)}{\phi(x)}$ is neither zero nor infinite, then from (A), we have $1 = \underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} \cdot \frac{\phi'(x)}{f'(x)}$

i.e. $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{\phi'(x)}$

Case II. If $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)}$ is zero i.e. if $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} = 0$

then $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} + 1 = 1$

i.e $\underset{x\to a}{Lt} \frac{f(x)+\phi(x)}{\phi(x)} = 1$

which is not zero, hence by case I, we have

$$\underset{x\to a}{Lt} \frac{f(x)+\phi(x)}{\phi(x)} = \underset{x\to a}{Lt} \frac{f'(x)+\phi'(x)}{\phi'(x)}$$

i.e. $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} + 1 = \underset{x\to a}{Lt} \frac{f'(x)}{\phi'(x)} + 1$

i.e. $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{\phi'(x)}$

Case III If $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} = \infty$, then

$\underset{x\to a}{Lt} \frac{\phi(x)}{f(x)} = 0$ and hence by case II, we have

$$\underset{x\to a}{Lt} \frac{\phi(x)}{f(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{\phi'(x)}$$

i.e. $\underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{\phi'(x)}$

Hence the theorem is true in all cases.

<u>Remark</u> : Other Indeterminate forms can also be evaluated by bringing them in $\frac{0}{0}$ form and proceeding in the same manner as above.

Examples

Ex. 1.

Evaluate :- $\underset{x\to 0}{Lt} \dfrac{x-\tan x}{x^3}$

Ans:

$$\underset{x\to 0}{Lt} \frac{x-\tan x}{x^3} \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{1-\sec^2 x}{3x^2} \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{-2\sec x \cdot \sec x \tan x}{6x} = \underset{x\to 0}{Lt} \frac{-2\tan x \sec^2 x}{6x} \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{-2\sec^4 x - 4\sec^2 x \tan^2 x}{6}$$

$$= \frac{-2(1)^4 - 4(1)^2(0)}{6} = \frac{-2}{6} = \frac{-1}{3}$$

Ex 2. Find the value of $\underset{x\to 0}{Lt} x^{\sin x}$ it is of the form 0^0 as $x\to 0$.

Ans: Let $y = x^{\sin x}$, Taking log of both sides, we get

$$\log y = \log x^{\sin x} = \sin x \log x \quad \text{——— (1)}$$

The right side of (1) takes the form $0 \times \infty$

Now, $\underset{x\to 0}{Lt} \sin x \log x = \underset{x\to 0}{Lt} \dfrac{\log x}{\frac{1}{\sin x}} \left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \underset{x\to 0}{Lt} \frac{\log x}{\operatorname{cosec} x} \left[\frac{\infty}{\infty} \text{ form}\right] = \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} = \frac{\sin^2 x}{x\cos x} \left[\frac{0}{0} \text{ form}\right]$$

$$= \frac{\sin 2x}{\cos x - x\sin x} = \frac{-0}{1-0} = 0$$

$\therefore \log y = 0$ Hence $y = e^0 = 1$, $\therefore \underset{x\to 0}{Lt} x^{\sin x} = 1$

Again $\log y = \underset{x\to 0}{Lt} \log (\cos x)^{\cot^2 x}$

$$= \underset{x\to 0}{Lt} \cot^2 x \log(\cos x) \, [\infty \times 0 \text{ form}] \quad \text{——— (1)}$$

$$= \underset{x\to 0}{Lt} \frac{\log(\cos x)}{\tan^2 x} \left[\frac{0}{0} \text{ form}\right] = \underset{x\to 0}{Lt} \frac{\frac{-\sin x}{\cos x}}{2\tan x \sec^2 x}$$

$$= \underset{x\to 0}{Lt} \frac{-\tan x}{2\tan x \sec^2 x} = \underset{x\to 0}{Lt} -\frac{1}{2\sec^2 x} = -\frac{1}{2}$$

$\therefore \log y = -\frac{1}{2}$ by taking antilog we have

$$y = e^{-\frac{1}{2}} = \frac{1}{e^{\frac{1}{2}}} = \frac{1}{\sqrt{e}}$$